

# Gauss-Bonnet Black Holes in dS Spaces

Ron-Gen Cai \*

*Institute of Theoretical Physics, Chinese Academy of Sciences,  
P.O. Box 2735, Beijing 100080, China*

*Interdisciplinary Center for Theoretical Study, University of Science  
and Technology of China, Hefei, Anhui 230026, China*

Qi Guo †

*Institute of Theoretical Physics, Chinese Academy of Sciences,  
P.O. Box 2735, Beijing 100080, China*

## Abstract

We study the thermodynamic properties associated with black hole horizon and cosmological horizon for the Gauss-Bonnet solution in de Sitter space. When the Gauss-Bonnet coefficient is positive, a locally stable small black hole appears in the case of spacetime dimension  $d = 5$ , the stable small black hole disappears and the Gauss-Bonnet black hole is always unstable quantum mechanically when  $d \geq 6$ . On the other hand, the cosmological horizon is found always locally stable independent of the spacetime dimension. But the solution is not globally preferred, instead the pure de Sitter space is globally preferred. When the Gauss-Bonnet coefficient is negative, there is a constraint on the value of the coefficient, beyond which the gravity theory is not well defined. As a result, there is not only an upper bound on the size of black hole horizon radius at which the black hole horizon and cosmological horizon coincide with each other, but also a lower bound depending on the Gauss-Bonnet coefficient and spacetime dimension. Within the physical phase space, the black hole horizon is always thermodynamically unstable and the cosmological horizon is always stable, further, as the case of the positive coefficient, the pure de Sitter space is still globally preferred. This result is consistent with the argument that the pure de Sitter space corresponds to an UV fixed point of dual field theory.

---

\*Email address: cairg@itp.ac.cn

†Email address: guoqi@itp.ac.cn

## I. INTRODUCTION

Higher derivative curvature terms naturally occur in many occasions, such as in the quantum field theory in curved space [1] and in the effective low-energy action of string theories. In the latter case, due to the AdS/CFT correspondence [2], these terms can be viewed as the corrections of large  $N$  expansion of boundary CFTs in the side of dual field theory. In the side of gravity, however, because of the nonlinearity of Einstein equations, it is quite difficult to find nontrivially exact analytical solutions of the Einstein equations with these higher derivative terms. In most cases, one has to adopt some approximation methods or find solutions numerically.

Up to the quadratic curvature terms, there is a special composition,

$$\mathcal{L}_{\text{GB}} = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \quad (1.1)$$

which is often called the Gauss-Bonnet term. The Einstein gravity with the Gauss-Bonnet term has some remarkable features in some sense. For instance, the resulting equations of motion have no more than second derivatives of metric and the theory has been shown to be free of ghosts when it is expanded about the flat space, evading any problems with unitarity [3]. Further, it has been argued that the Gauss-Bonnet term appears as the leading correction [4] to the effective low-energy action of the heterotic string theory. In addition, it has already been found that exact analytical solutions with spherical symmetry can be obtained in this gravity theory [3,5–7].

The thermodynamics and geometric structure of the Gauss-Bonnet black hole in asymptotically flat space have been analyzed in Refs. [8,9]. In a previous paper [7]<sup>1</sup>, we studied the thermodynamics and phase structure of topological black holes in Einstein gravity with the Gauss-Bonnet term and a negative cosmological constant. Those topological black holes are asymptotically anti-de Sitter (AdS) and their event horizon can be a hypersurface with positive, zero, or negative constant curvature. In the present paper, we will study the properties of Gauss-bonnet black holes in asymptotically de Sitter (dS) space. Studying the Gauss-Bonnet black hole in dS space is of interest in its own right. On the other hand, we hope to gain some insights into the dual field theory in the sense of the dS/CFT correspondence [12].

It is well-known that unlike the cases of asymptotically flat space and asymptotically AdS space, it is not an easy matter to calculate conserved charges associated with an asymptotically dS space because of the absence of spatial infinity and a globally timelike Killing vector in such a spacetime. On the other hand, there is a cosmological event horizon, except for the black hole horizon, for the spacetime of black holes in dS space. Like the black hole horizon, there is also a thermodynamic feature for the cosmological horizon [13]. In general the Hawking temperatures associated with the black hole horizon and cosmological horizon, respectively, are not equal; therefore the spacetime for black hole in dS space is unstable quantum mechanically.

---

<sup>1</sup>The thermodynamics and phase structure of black hole solutions perturbed by quadratic curvature terms in asymptotically AdS space has also been discussed in Refs. [10], see also [6] for the case of black holes in the dimensionally continued gravity.

In this paper we will discuss separately the thermodynamics of black hole horizon and cosmological horizon. Namely, we view the black hole horizon and cosmological horizon as two thermodynamic systems. For the case of black hole horizon, we calculate the black hole mass in the definition due to Abbott and Deser (AD) [11], by considering the deviation of metric from the pure dS space being defined as the vacuum (lowest energy state)<sup>2</sup>. In terms of this definition, the gravitational mass of asymptotically dS space is always positive, and coincides with the ADM mass in asymptotically flat space, when the cosmological constant goes to zero. For the case of cosmological horizon, we will adopt the prescription due to Balasubramanian, de Boer and Minic (BBM) [16]. In this prescription, except for a constant, which depends on the cosmological constant and space dimension and can be regarded as the Casimir energy of the dual field theory in the spirit of the dS/CFT correspondence, the gravitational mass is just the AD mass, but with an opposite sign [16–18,20]. The BBM mass is measured at the far past ( $\mathcal{I}^-$ ) or far further ( $\mathcal{I}^+$ ) boundary of dS space, which is outside the cosmological horizon. With these definitions, thermodynamic quantities associated with the black hole horizon and cosmological horizon obey the first law of thermodynamics, respectively. In Refs. [19] we have also shown they satisfy respectively the Cardy-Verlinde formula this way. In particular, it was argued [21] that for the Euclidean black hole-de Sitter geometry which is closely related to the horizon thermodynamics, when deals with the thermodynamics of one of two horizons, one should view the other as the boundary. In this way, one has well-defined Hamiltonians associated with the black hole horizon and cosmological horizon, respectively. Therefore the point of viewing black hole horizon and cosmological horizon as two thermodynamic systems should be reasonable.

The organization of the paper is as follows. In the next section we present the solution of the Gauss-Bonnet black hole in dS space. In Sec. III and IV we discuss the thermodynamics and phase structure of black hole horizon and cosmological horizon, respectively. This paper is ended in Sec. V with some conclusions and discussions.

## II. GAUSS-BONNET BLACK HOLE SOLUTION IN DE SITTER SPACE

We start with the Einstein-Hilbert action with the Gauss-Bonnet term (1.1) and a positive cosmological constant,  $\Lambda = (d-1)(d-2)/2l^2$ , in  $d$  dimensions

$$S = \frac{1}{16\pi G} \int d^d x \sqrt{-g} \left( R - \frac{(d-1)(d-2)}{l^2} + \alpha \mathcal{L}_{\text{GB}} \right), \quad (2.1)$$

where  $G$  is the Newton constant and  $\alpha$  is the Gauss-Bonnet coefficient with dimension  $(\text{length})^2$ . From this action we obtain the equations of motion

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = & -\frac{(d-1)(d-2)}{2l^2}g_{\mu\nu} + \alpha \left( \frac{1}{2}g_{\mu\nu}(R_{\gamma\delta\lambda\sigma}R^{\gamma\delta\lambda\sigma} - 4R_{\gamma\delta}R^{\gamma\delta} + R^2) \right. \\ & \left. - 2RR_{\mu\nu} + 4R_{\mu\gamma}R^\gamma_\nu + 4R_{\gamma\delta}R^\delta_{\mu\nu} - 2R_{\mu\gamma\delta\lambda}R^\gamma_{\nu}{}^{\delta\lambda} \right). \end{aligned} \quad (2.2)$$

---

<sup>2</sup>In Ref. [11] the authors consider the Einstein gravity with a cosmological constant. When higher derivative curvature terms are present, similar mass definition of gravitational field has been discussed in Ref. [14](see also discussions for Gauss-Bonnet gravity in Ref. [15]).

For the metric we adopt the following ansatz of spherical symmetry

$$ds^2 = -e^{2\nu} dt^2 + e^{2\lambda} dr^2 + r^2 d\Omega_{d-2}^2, \quad (2.3)$$

where  $\nu$  and  $\lambda$  are functions of  $r$  only, and  $d\Omega_{d-2}^2$  represents the line element of a  $(d-2)$ -dimensional unit sphere with volume  $\Omega_{d-2} = 2\pi^{(d-1)/2}/\Gamma[(d-1)/2]$ . To find a solution with metric (2.3), there is a simple method [3]: substituting the metric ansatz (2.3) into the action (2.1) yields

$$S = \frac{(d-2)\Omega_{d-2}}{16\pi G} \int dt dr e^{\nu+\lambda} \left[ r^{d-1} \varphi(1 + \tilde{\alpha}\varphi) - \frac{r^{d-1}}{l^2} \right]', \quad (2.4)$$

where a prime denotes the derivative with respect to  $r$ ,  $\tilde{\alpha} = \alpha(d-3)(d-4)$  and  $\varphi = r^{-2}(1 - e^{-2\lambda})$ . From the action one has

$$\begin{aligned} e^{\nu+\lambda} &= 1, \\ \varphi(1 + \tilde{\alpha}\varphi) - \frac{1}{l^2} &= \frac{16\pi G M}{(d-2)\Omega_{d-2} r^{d-1}}. \end{aligned} \quad (2.5)$$

Then one obtains the exact solution

$$e^{2\nu} = e^{-2\lambda} = 1 + \frac{r^2}{2\tilde{\alpha}} \left( 1 \mp \sqrt{1 + \frac{64\pi G \tilde{\alpha} M}{(d-2)\Omega_{d-2} r^{d-1}} + \frac{4\tilde{\alpha}}{l^2}} \right), \quad (2.6)$$

where  $M$  is an integration constant, which is just the AD mass of the solution. This exact solution was first found by Boulware and Deser in Ref. [3]. In [7] we extended this solution to the case where the unit sphere  $d\Omega_{d-2}$  is replaced by a hypersurface with positive, zero or negative constant curvature.

Note that the solution (2.6) has a singularity at  $r = 0$  if  $\tilde{\alpha} > 0$ . When  $\tilde{\alpha} < 0$  there is an additional singularity at the place where the square root vanishes in (2.6). In addition note that there are two branches in the solution (2.6) with “−” and “+” signs, respectively. When the integration constant,  $M$ , vanishes, the solution reduces to

$$e^{2\nu} = e^{-2\lambda} = 1 + \frac{r^2}{2\tilde{\alpha}} \left( 1 \mp \sqrt{1 + \frac{4\tilde{\alpha}}{l^2}} \right). \quad (2.7)$$

This is a dS or AdS solution depending on the effective curvature radius,

$$\frac{1}{l_{\text{eff}}^2} = -\frac{1}{2\tilde{\alpha}} \left( 1 \mp \sqrt{1 + \frac{4\tilde{\alpha}}{l^2}} \right). \quad (2.8)$$

When  $\tilde{\alpha} > 0$ , one has  $l_{\text{eff}}^2 > 0$  for the branch with the “−” sign, while  $l_{\text{eff}}^2 < 0$  for the “+” sign. Therefore, in this case, the solution is asymptotically dS for the branch with the “−” sign, and asymptotically AdS for the sign “+”, although the cosmological constant,  $\Lambda$ , in the action (2.1) is positive. On the other hand, when  $\tilde{\alpha} < 0$ , one has  $l_{\text{eff}}^2 > 0$  for both branches, which means the solution is always asymptotically dS. But in that case, one can see from (2.8) that the Gauss-Bonnet parameter has to satisfy

$$\tilde{\alpha}/l^2 \geq -1/4, \quad (2.9)$$

for the branch with the “−” sign. Otherwise, the theory is not well defined. Here it should be stressed that the constraint (2.9) is obtained from the vacuum solution of the theory (2.1). To avoid a naked singularity, the more stringent constraint will be (3.19), as discussed below.

That the solution (2.6) has two branches implies that the theory has two different vacua (2.7). In Ref. [3] Boulware and Deser have shown that the branch with the “+” sign is unstable; the graviton propagating on the background in this branch is ghost, while the branch with the “−” sign is stable and the graviton is free of ghost. The branch with the “+” sign is of less physical interest. Therefore we will not discuss this branch and focus on the branch with the “−” sign in what follows.

### III. THERMODYNAMICS OF THE BLACK HOLE HORIZON

One can see from (2.7) that when  $M = 0$ , there is a cosmological horizon at  $r_c = l_{\text{eff}}$ . When  $M$  increases from zero, like the case of the Schwarzschild-dS solution, a black hole horizon appears in the solution (2.6) and the cosmological horizon shrinks. That is, in general there are two positive real roots for the equation,  $e^{2\nu} = 0$ . The large one is the cosmological horizon  $r_c$  and the small one is the black hole horizon  $r_+$ . In this section we first discuss the thermodynamics associated with the black hole horizon.

In terms of the black hole horizon, the mass of the Gauss-Bonnet black hole, namely the AD mass of the solution, can be expressed as

$$M = \frac{(d-2)\Omega_{d-2}r_+^{d-3}}{16\pi G} \left( 1 + \frac{\tilde{\alpha}}{r_+^2} - \frac{r_+^2}{l^2} \right). \quad (3.1)$$

Obviously, when  $\tilde{\alpha} = 0$ , this quantity reduces to the mass of the Schwarzschild-dS black hole in  $d$  dimensions. The Hawking temperature associated with the black hole horizon can easily be obtained by requirement of the absence of conical singularity at the black hole horizon in the Euclidean sector of the Gauss-Bonnet black hole solution in dS space. It turns out

$$T = \frac{(d-5)\tilde{\alpha} + (d-3)r_+^2 - (d-1)r_+^4/l^2}{4\pi r_+(r_+^2 + 2\tilde{\alpha})}. \quad (3.2)$$

Another important thermodynamic quantity is the entropy of black hole horizon. In Einstein gravity, entropy of black hole satisfies the so-called area formula [22]. Namely the entropy is equal to one-quarter of the horizon area. When higher derivative curvature terms are present, however, this statement no longer holds. Wald has shown that entropy of black hole in any gravity theory is always a function of horizon geometry [23]. From Refs. [7,8] we can read the entropy of the Gauss-Bonnet black hole in dS space

$$S = \frac{\Omega_{d-2}r_+^{d-2}}{4G} \left( 1 + \frac{2(d-2)\tilde{\alpha}}{(d-4)r_+^2} \right), \quad (3.3)$$

since the cosmological constant does not explicitly occur in this expression. Indeed we can show that three thermodynamic quantities (3.1), (3.2) and (3.3) obey the first law of thermodynamics,  $dM = TdS$ .

The quantity indicating the local stability of black hole is the heat capacity. For the Gauss-Bonnet black hole in dS space, it is

$$C \equiv \left( \frac{\partial M}{\partial T} \right) = \left( \frac{\partial M}{\partial r_+} \right) \left( \frac{\partial r_+}{\partial T} \right), \quad (3.4)$$

where

$$\begin{aligned} \frac{\partial M}{\partial r_+} &= \frac{(d-2)\Omega_{d-2}}{4G} r_+^{d-5} (r_+^2 + 2\tilde{\alpha}) T, \\ \frac{\partial T}{\partial r_+} &= \frac{1}{4\pi l^2 r_+^2 (r_+^2 + 2\tilde{\alpha})^2} [-(d-1)r_+^6 - (d-3)l^2 r_+^4 - 6(d-1)\tilde{\alpha} r_+^4 \\ &\quad + 2(d-3)\tilde{\alpha} l^2 r_+^2 - 3(d-5)\tilde{\alpha} l^2 r_+^2 - 2(d-5)\tilde{\alpha}^2 l^2]. \end{aligned} \quad (3.5)$$

By definition,  $F = M - TS$ , the free energy of the black hole is

$$\begin{aligned} F &= \frac{\Omega_{d-2} r_+^{d-5}}{16\pi G (d-4) l^2 (r_+^2 + 2\tilde{\alpha})} [(d-4)r_+^6 + (d-4)l^2 r_+^4 \\ &\quad + 6(d-2)\tilde{\alpha} r_+^4 + (d-8)\tilde{\alpha} l^2 r_+^2 + 2(d-2)\tilde{\alpha}^2 l^2]. \end{aligned} \quad (3.6)$$

Now we are in a position to discuss the thermodynamic stability and phase structure of the black hole.

(1) Let us first consider the case of  $\tilde{\alpha} > 0$ , which is the case of the heterotic string theory [4]. When the Hawking temperature (3.2) vanishes, we obtain

$$r_+^2 = r_{1,2}^2 = \frac{(d-3)l^2}{2(d-1)} \left( 1 \pm \sqrt{1 + \frac{(d-1)(d-5)}{(d-3)^2} \frac{4\tilde{\alpha}}{l^2}} \right), \quad (3.7)$$

from which we see that when  $d = 5$  only, there are two real positive roots: one is  $r_+ = r_2 = 0$ , the other is  $r_+^2 = r_1^2 = l^2/2$ . The large one  $r_1$  is the horizon radius of maximal black hole in the solution (2.6), beyond which the singularity behind the black hole horizon becomes naked. The maximal black hole with radius  $r_1$  in (3.7) is therefore the counterpart of the Nariai black hole in the Gauss-Bonnet gravity, there the black hole horizon and cosmological horizon coincide with each other and therefore the Hawking temperature is zero.

When  $d = 5$ , the inverse temperature starts from infinity at  $r_+ = 0$ , reaches a minimal value at some place and then goes to infinity again at the maximal black hole horizon radius. This behavior can be seen from the heat capacity (3.5). When the black hole horizon satisfies

$$0 < r_+^2 < r_0^2 = \frac{l^2}{4} \left( 1 + \frac{12\tilde{\alpha}}{l^2} \right) \left( \sqrt{1 + \frac{16\tilde{\alpha}}{l^2} \left( 1 + \frac{12\tilde{\alpha}}{l^2} \right)^{-2}} - 1 \right), \quad (3.8)$$

the heat capacity is positive and it becomes negative for  $r_0^2 < r_+^2 < l^2/2$ . Here  $l/\sqrt{2}$  is the maximal black hole horizon radius in the case of five dimensions. This behavior is quite different from the case without the Gauss-Bonnet term, there the heat capacity of the black hole horizon is always negative. Therefore the small Gauss-Bonnet black hole satisfying (3.8) is thermodynamically stable. Here there is no restriction on the Gauss-Bonnet coefficient, except for the positivity of the coefficient.

When  $d \geq 6$ , the equation  $T = 0$  has only one real positive root  $r_1$  in (3.7), the other is negative, without any physical meaning. One can see from (3.2) that the inverse temperature always starts from zero at  $r_+ = 0$  and goes to infinity monotonically at the maximal horizon radius  $r_1$  in (3.7), which implies that the heat capacity is always negative in this case. This indicates the instability of the Gauss-bonnet black hole. When  $d \geq 6$ , therefore the thermodynamics properties of the Gauss-Bonnet black hole in dS space is qualitatively similar to those of Schwarzschild-dS black hole, the black hole in dS space without the Gauss-Bonnet term. Thus the thermodynamics of the black hole horizon becomes remarkably related to the spacetime dimension. In Fig. 1 we plot the inverse temperature versus the radius of black hole horizon.

Checking the free energy (3.6), however, we find that it is always positive whatever the spacetime dimension and the Gauss-Bonnet coefficient are. In Fig. 2 the free energy of a five-dimensional Gauss-Bonnet black hole is plotted versus the horizon radius and the Gauss-Bonnet coefficient. In Fig. 3 we plot the free energy versus the horizon radius and spacetime dimension with a fixed Gauss-Bonnet coefficient. The positivity of the free energy implies that the black hole solution is not globally preferred, instead the dS space (2.7) is globally preferred since we have taken the dS space as the vacuum state.

(2) When  $\tilde{\alpha} < 0$ , from the solution (2.6) we find that the black hole horizon must satisfies

$$r_+^2 \geq -2\tilde{\alpha}. \quad (3.9)$$

However, from the entropy formula (3.3) one can see when

$$-2\tilde{\alpha} \leq r_+^2 < -\frac{d-2}{d-4}2\tilde{\alpha}, \quad (3.10)$$

the entropy of black hole horizon is negative, which should be ruled out in the physical phase space since a negative entropy is meaningless. Therefore we obtain a constraint on the minimal horizon radius of the Gauss-Bonnet black hole

$$r_+^2 \geq -2\tilde{\alpha}\frac{d-2}{d-4}, \quad (3.11)$$

when  $\tilde{\alpha} < 0$ . As the above equation holds, the black hole has vanishing entropy.

In this case, when  $d = 5$ , from the temperature (3.2), we find that the horizon radius falls into the range

$$-6\tilde{\alpha} \leq r_+^2 \leq l^2/2. \quad (3.12)$$

As the case of  $\tilde{\alpha} > 0$ , here  $r_+ = l/\sqrt{2}$  is the horizon radius of the maximal black hole, there both the black hole horizon and cosmological horizon coincide with each other and the Hawking temperature vanishes. From (3.12) we see that there is a more stringent constraint than the one (2.9):

$$\tilde{\alpha}/l^2 \geq -1/12. \quad (3.13)$$

Further, one can see from (3.5) that the heat capacity changes its behavior at the place

$$r_+^2 = \frac{l^2}{4} \left( 1 + \frac{12\tilde{\alpha}}{l^2} \right) \left( -1 \pm \sqrt{1 + \frac{16\tilde{\alpha}}{l^2} \left( 1 + \frac{12\tilde{\alpha}}{l^2} \right)^{-2}} \right). \quad (3.14)$$

In order the equation (3.14) to have real root, one has to have

$$\tilde{\alpha}/l^2 \leq -1/12, \quad (3.15)$$

which contradicts the condition (3.13). This means that the inverse temperature always starts monotonically from a finite value at the minimal radius given by (3.12) to infinity at the maximal radius  $l/\sqrt{2}$  given in (3.12). This indicates that when  $\tilde{\alpha} < 0$ , the five-dimensional Gauss-Bonnet black hole in dS space has a negative heat capacity and then it is unstable as the case without the Gauss-Bonnet term. The inverse temperature of the black hole horizon, plotted in Fig. 4, shows this fact.

When  $d \geq 6$ , the condition that the Hawking temperature (3.2) vanishes is

$$r_+^2 = r_{3,4}^2 = \frac{(d-3)l^2}{2(d-1)} \left( 1 \pm \sqrt{1 + \frac{(d-1)(d-5)}{(d-3)^2} \frac{4\tilde{\alpha}}{l^2}} \right). \quad (3.16)$$

To have two positive real roots, one has

$$\frac{\tilde{\alpha}}{l^2} > -\frac{(d-3)^2}{4(d-1)(d-5)}. \quad (3.17)$$

The large root corresponds to the maximal Gauss-Bonnet black hole in dS space. But the small one is outside the constraint (3.11). Therefore, the behavior of the inverse temperature is similar to the case of  $d = 5$ : it starts from a finite value at the minimal black hole horizon given by (3.11) and goes to infinity at the maximal radius given (3.16) monotonically. As a result, the  $d \geq 6$  Gauss-Bonnet black hole in dS space is also unstable when  $\tilde{\alpha} < 0$ , like the case without the Gauss-Bonnet term. In Fig. 5 we plot the inverse temperature of the black hole in seven dimensions with a fixed Gauss-Bonnet coefficient. Note that the value of the side of right hand in (3.17) is smaller than  $-1/4$  given in (2.9). Therefore it seems that the true constraint on the coefficient  $\tilde{\alpha}$  is (2.9), rather than (3.17). It turns out this is not correct. The reason is that since the horizon radius falls into the range

$$-2\tilde{\alpha} \frac{d-2}{d-4} \leq r_+^2 \leq r_3^2, \quad (3.18)$$

which gives us a more stringent constraint

$$\frac{\tilde{\alpha}}{l^2} \geq -\frac{(d^2 - d - 8)(d - 4)}{4(d - 1)(d - 2)^2}. \quad (3.19)$$

We have checked numerically that within the ranges (3.18) and (3.19), the heat capacity is always negative, while the free energy of the black hole horizon is always positive as the case of  $\tilde{\alpha} > 0$ <sup>3</sup>.

---

<sup>3</sup>In the range  $-2\tilde{\alpha} < r_+^2 < -2\tilde{\alpha}(d-2)/(d-4)$ , a stable small black hole with positive heat capacity may appear, but it has a negative entropy. As a result, it should be ruled out in the physical phase space, the true constraint on the horizon radius is given by (3.18).

#### IV. THERMODYNAMICS OF THE COSMOLOGICAL HORIZON

For the cosmological horizon denoted by  $r_c$ , the associated Hawking temperature  $T_c$  is

$$T_c = \frac{-(d-5)\tilde{\alpha} - (d-3)r_c^2 + (d-1)r_c^4/l^2}{4\pi r_c(r_c^2 + 2\tilde{\alpha})}. \quad (4.1)$$

and entropy  $S_c$

$$S_c = \frac{\Omega_{d-2}r_c^{d-2}}{4G} \left( 1 + \frac{2(d-2)\tilde{\alpha}}{(d-4)r_c^2} \right). \quad (4.2)$$

The thermodynamic energy associated with the cosmological horizon can be calculated using the BBM prescription [16] (namely, the surface counterterm approach). In this prescription, it has been found that the BBM mass for black holes in dS spaces in Einstein theory is just the negative AD mass, see, for example, Refs. [16–18,21], except for a constant, which is not relevant to the present discussion. For the Gauss-Bonnet black holes in dS space, the BBM prescription is also applicable. As the case of Einstein gravity, it turns out that the thermodynamic energy of the cosmological horizon is the negative AD mass (see also [20,10]),

$$E = -M = -\frac{(d-2)\Omega_{d-2}r_c^{d-3}}{16\pi G} \left( 1 + \frac{\tilde{\alpha}}{r_c^2} - \frac{r_c^2}{l^2} \right). \quad (4.3)$$

A self-consistency check is that these three thermodynamic quantities obey the first law of thermodynamics,  $dE = T_c dS_c$ . To see the thermodynamic stability, we calculate the heat capacity of the cosmological horizon

$$C_c \equiv \left( \frac{\partial E}{\partial T_c} \right) = \left( \frac{\partial E}{\partial r_c} \right) \left( \frac{\partial r_c}{\partial T_c} \right), \quad (4.4)$$

where

$$\begin{aligned} \left( \frac{\partial E}{\partial r_c} \right) &= \frac{(d-2)\Omega_{d-2}}{4G} r_c^{d-5} (r_c^2 + 2\tilde{\alpha}) T_c, \\ \frac{\partial T_c}{\partial r_c} &= \frac{1}{4\pi l^2 r_c^2 (r_c^2 + 2\tilde{\alpha})^2} [(d-1)r_c^6 + (d-3)l^2 r_c^4 + 6(d-1)\tilde{\alpha} r_c^4 \\ &\quad - 2(d-3)\tilde{\alpha} l^2 r_c^2 + 3(d-5)\tilde{\alpha} l^2 r_c^2 + 2(d-5)\tilde{\alpha}^2 l^2]. \end{aligned} \quad (4.5)$$

And the free energy,  $F_c = E - T_c S_c$ , is

$$\begin{aligned} F_c &= \frac{\Omega_{d-2}r_c^{d-5}}{16\pi G(d-4)l^2(r_c^2 + 2\tilde{\alpha})} [-(d-4)r_c^6 - (d-4)l^2 r_c^4 \\ &\quad - 6(d-2)\tilde{\alpha} r_c^4 - (d-8)\tilde{\alpha} l^2 r_c^2 - 2(d-2)\tilde{\alpha}^2 l^2]. \end{aligned} \quad (4.6)$$

The cosmological horizon radius has a range in size: the minimal value is just the maximal black hole horizon  $r_3$  in (3.16) discussed in the previous section, while the maximal radius

is  $l_{\text{eff}}$  given in (2.8), there the integration constant  $M$  vanishes. Namely the cosmological horizon is in the following region

$$r_3^2 \leq r_c^2 \leq l_{\text{eff}}^2. \quad (4.7)$$

Within this region, it is easy to show that the heat capacity (4.5) is positive and therefore the inverse temperature always starts from infinity, where the cosmological horizon coincides with the black hole horizon, and monotonically goes to a finite value, which corresponds to the inverse temperature of the vacuum dS space (2.7). This implies that the thermodynamics of the cosmological horizon is locally stable. From (4.6) we see that the free energy is always negative. But this does not mean that the solution is globally preferred since when we calculate the gravitational mass, the pure dS space (2.7) is regarded as the vacuum. This vacuum has zero AD mass, but has non-zero Hawking temperature and entropy associated with the cosmological horizon

$$T_c^{\text{vac}} = \frac{1}{2\pi l_{\text{eff}}}, \quad S_c^{\text{vac}} = \frac{\Omega_{d-2} l_{\text{eff}}^{d-2}}{4G} \left( 1 + \frac{2(d-2)\tilde{\alpha}}{(d-4)l_{\text{eff}}^2} \right). \quad (4.8)$$

And the corresponding free energy is

$$F_c^{\text{vac}} = -\frac{\Omega_{d-2} l_{\text{eff}}^{d-3}}{8\pi G} \left( 1 + \frac{2(d-2)\tilde{\alpha}}{(d-4)l_{\text{eff}}^2} \right). \quad (4.9)$$

To see whether or not the solution with non-vanishing  $M$  is globally preferred, we have to compare the two free energies (4.6) and (4.9):

$$\Delta F = F_c - F_c^{\text{vac}}. \quad (4.10)$$

If  $\Delta F > 0$ , the solution with non-vanishing  $M$  is not globally preferred, otherwise it is preferred. It seems difficult to analytically prove  $\Delta F > 0$ , but we have checked numerically that indeed  $\Delta F > 0$  within the range (4.7). Therefore the pure dS space (2.7) is globally preferred. Namely although the thermodynamics of the cosmological horizon is locally stable, it will decay to the pure dS space. In Fig. 6 we plot the difference of the two free energies,  $\Delta F$ , versus the Gauss-Bonnet coefficient and the cosmological horizon radius in the case of five dimensions. The case of ten dimensions is plotted in Fig. 7.

When  $\tilde{\alpha} < 0$ , the coefficient has to satisfy the constraint (3.19), again. Within the horizon range (4.7) and the constraint (3.19), we have numerically checked that the difference of the free energies associated with the cosmological horizon is always positive as the case  $\tilde{\alpha} > 0$  (As an example, we plot the free energy difference associated with the cosmological horizon in the case of five dimensions in Fig. 8.). As a result, the pure de Sitter space (2.7) is globally preferred again.

## V. CONCLUSION AND DISCUSSION

In summary we have discussed the thermodynamic properties and phase structures associated with the black hole horizon and cosmological horizon for the Gauss-Bonnet black hole-de Sitter spacetime. The black hole horizon and cosmological horizon are viewed as two

thermodynamic systems. When the Gauss-Bonnet coefficient is positive, which is the case for the effective low energy action of the heterotic string theory, a locally stable small black hole whose radius satisfying (3.8) appears in  $d = 5$  dimensions, which is absent in the case without the Gauss-Bonnet term. When the spacetime dimension  $d \geq 6$ , the stable small black hole disappears; the black hole is always unstable thermodynamically as the case without the Gauss-Bonnet term. Contrary to the black hole horizon, the cosmological horizon is always thermodynamically stable with positive heat capacity. But the Gauss-Bonnet black hole solution in de Sitter space is not globally preferred, instead the pure de Sitter space (2.7) is globally preferred, which has lower free energy than the case with nonvanishing  $M$ .

On the other hand, when the Gauss-Bonnet coefficient is negative, there is a bound on the coefficient given by (2.9), otherwise, the gravity theory is not well-defined (note that the constraint (2.9) is derived from the vacuum solution of the theory, it does not warrant that a naked singularity does not occur in this case. In fact, a true constraint is (3.19), under which a black hole solution is meaningful.). In this case, the horizon radius of the Gauss-Bonnet black hole has not only an upper bound  $r_3$  given by (3.16), there the black hole horizon coincides with the cosmological horizon, but also a lower bound. From the solution (2.6), the lower bound seems to be  $-2\tilde{\alpha}$  given by (3.9). Checking the entropy (3.3) of black hole horizon tells us that within the range (3.10), the entropy associated with the black hole horizon is negative. As a result this range (3.10) should be ruled out in the physical phase space. Therefore the true lower bound of the horizon radius is given by (3.11). Further it gives more stringent constraint on the value of the Gauss-Bonnet coefficient (3.19). Within the coefficient (3.19) and the horizon range (3.18), the black hole horizon becomes always thermodynamically unstable and the cosmological horizon is still thermodynamically stable, that is, in this case the stable small black hole in five dimensions disappears. Checking the free energies associated with the black hole horizon and cosmological horizon, respectively, reveals that the pure de Sitter space is still globally preferred.

Therefore, both thermodynamic discussions of black hole horizon and cosmological horizon lead to the same conclusion that a pure de Sitter space is globally preferred. This result is consistent with the argument that a pure de Sitter space corresponds to an UV fixed point of the renormalization group flow of the dual field theory in the dS/CFT correspondence [24,16].

Finally we would like to stress that as argued in INTRODUCTION, although a black hole-de Sitter spacetime is unstable quantum mechanically because two Hawking temperatures associated with black hole horizon and cosmological horizon are in general not equal, except for the Nariai solution or its generalizations, where two temperatures equal to each other. So it is not an easy matter to study the thermodynamic properties of spacetime for a black hole in dS space as an entire. In particular, Teitelboim recently argued [21] that for the Euclidean black hole-de Sitter geometry which is closely related to the horizon thermodynamics, when deals with the thermodynamics of one of two horizons, one should view the other as the boundary. In this way, one has well-defined Hamiltonians associated with the black hole horizon and cosmological horizon, respectively. In this paper we just followed this spirit to discuss the thermodynamic properties associated with black hole horizon and cosmological horizon, respectively, and to obtain the conclusion that the pure de Sitter space is globally preferred and it is end point of decay. In addition, the local stability analysis of black hole horizon and cosmological horizon might be less motivated just due to different

temperatures. However, when two horizons separate with a very large distance, the effect of the Hawking evaporation of one horizon could be negligible on the other horizon. In this sense it might be of some interest and be of meaning to discuss local stability of two horizons, respectively. We wish that the present investigation together with a lot of existing literature concerning black hole-de Sitter spacetimes is in the way to completely understand classical and quantum properties of asymptotically de Sitter spaces.

### **ACKNOWLEDGMENTS**

This work was initiated while one of authors (R.G.C) was visiting the ICTS at USTC, whose hospitality is gratefully acknowledged. The research was supported in part by a grant from Chinese Academy of Sciences, a grant No. 10325525 from NSFC, and by the Ministry of Science and Technology of China under grant No. TG1999075401.

## REFERENCES

- [1] N. D. Birrell and P. C. Davies, “Quantum Fields In Curved Space,” Cambridge Univ. Press, 1982.
- [2] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1998)] [hep-th/9711200]; S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998) [hep-th/9802109]; E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].
- [3] D. G. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1985).
- [4] B. Zwiebach, Phys. Lett. B **156**, 315 (1985); R. I. Nepomechie, Phys. Rev. D **32**, 3201 (1985).
- [5] J. T. Wheeler, Nucl. Phys. B **268**, 737 (1986).
- [6] J. Crisostomo, R. Troncoso and J. Zanelli, Phys. Rev. D **62**, 084013 (2000) [arXiv:hep-th/0003271]; R. Aros, R. Troncoso and J. Zanelli, Phys. Rev. D **63**, 084015 (2001) [arXiv:hep-th/0011097].
- [7] R. G. Cai, Phys. Rev. D **65**, 084014 (2002) [arXiv:hep-th/0109133].
- [8] R. C. Myers and J. Z. Simon, Phys. Rev. D **38**, 2434 (1988).
- [9] D. L. Wiltshire, Phys. Rev. D **38**, 2445 (1988).
- [10] S. Nojiri and S. D. Odintsov, Phys. Lett. B **521**, 87 (2001) [Erratum-ibid. B **542**, 301 (2002)] [arXiv:hep-th/0109122]; S. Nojiri, S. D. Odintsov and S. Ogushi, Phys. Rev. D **65**, 023521 (2002) [arXiv:hep-th/0108172]; M. Cvetič, S. Nojiri and S. D. Odintsov, Nucl. Phys. B **628**, 295 (2002) [arXiv:hep-th/0112045]; Y. M. Cho and I. P. Neupane, Phys. Rev. D **66**, 024044 (2002) [arXiv:hep-th/0202140]; I. P. Neupane, Phys. Rev. D **67**, 061501 (2003) [arXiv:hep-th/0212092]; I. P. Neupane, arXiv:hep-th/0302132.
- [11] L. F. Abbott and S. Deser, Nucl. Phys. B **195**, 76 (1982).
- [12] A. Strominger, JHEP **0110**, 034 (2001) [arXiv:hep-th/0106113].
- [13] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).
- [14] S. Deser and B. Tekin, Phys. Rev. Lett. **89**, 101101 (2002) [arXiv:hep-th/0205318]; S. Deser and B. Tekin, Phys. Rev. D **67**, 084009 (2003) [arXiv:hep-th/0212292].
- [15] A. Padilla, Class. Quant. Grav. **20**, 3129 (2003) [arXiv:gr-qc/0303082]; J. P. Gregory and A. Padilla, Class. Quant. Grav. **20**, 4221 (2003) [arXiv:hep-th/0304250].
- [16] V. Balasubramanian, J. de Boer and D. Minic, Phys. Rev. D **65**, 123508 (2002) [arXiv:hep-th/0110108].
- [17] R. G. Cai, Y. S. Myung and Y. Z. Zhang, Phys. Rev. D **65**, 084019 (2002) [arXiv:hep-th/0110234].
- [18] A. M. Ghezelbash and R. B. Mann, JHEP **0201**, 005 (2002) [arXiv:hep-th/0111217].
- [19] R. G. Cai, Phys. Lett. B **525**, 331 (2002) [arXiv:hep-th/0111093]; R. G. Cai, Nucl. Phys. B **628**, 375 (2002) [arXiv:hep-th/0112253].
- [20] M. Cvetič, S. Nojiri and S. D. Odintsov, Nucl. Phys. B **628**, 295 (2002) [arXiv:hep-th/0112045].
- [21] C. Teitelboim, arXiv:hep-th/0203258; A. Gomberoff and C. Teitelboim, Phys. Rev. D **67**, 104024 (2003) [arXiv:hep-th/0302204].
- [22] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).
- [23] R. M. Wald, Phys. Rev. D **48**, 3427 (1993) [arXiv:gr-qc/9307038].
- [24] A. Strominger, JHEP **0111**, 049 (2001) [arXiv:hep-th/0110087].

## FIGURES

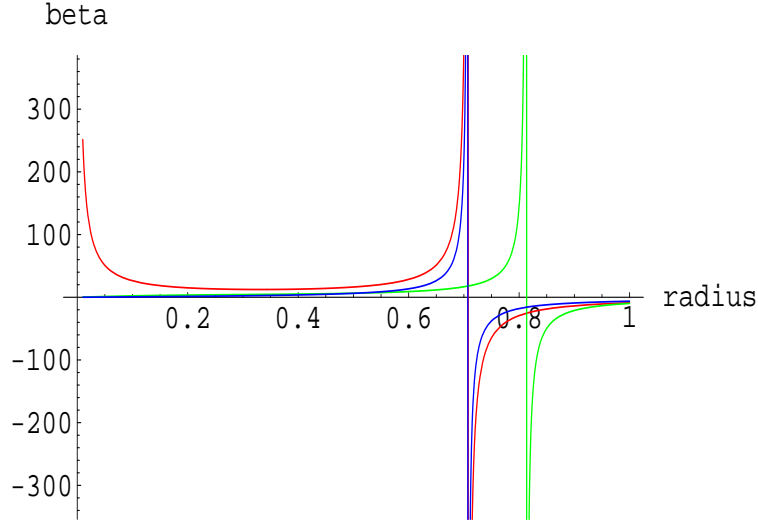


FIG. 1. The inverse temperature of the Gauss-Bonnet black holes in dS space. The red curve corresponds to the case of  $d = 5$  and  $\tilde{\alpha}/l^2 = 0.2$ , the blue one to the case of  $d = 5$  and  $\tilde{\alpha}/l^2 = 0$ , and the green one to the case of  $d = 6$  and  $\tilde{\alpha}/l^2 = 0.2$ . Note that the region with negative temperature should be ruled out in the physical phase space.

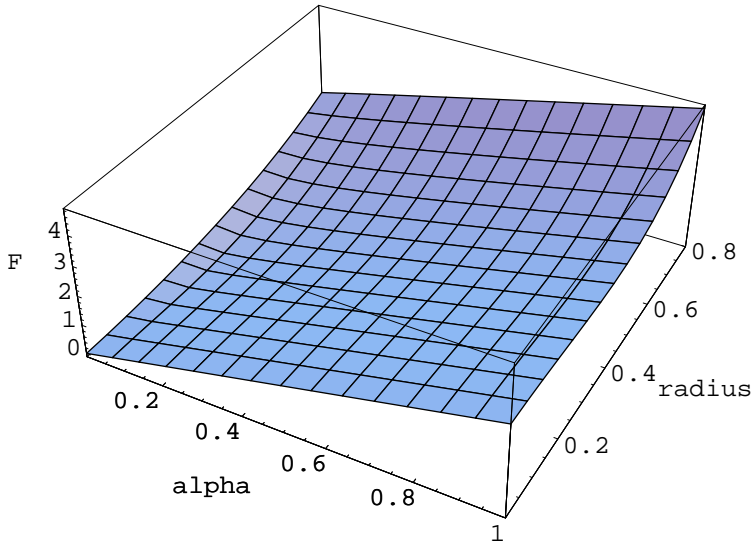


FIG. 2. The free energy of the five dimensional Gauss-Bonnet black hole versus the Gauss-Bonnet coefficient and horizon radius.

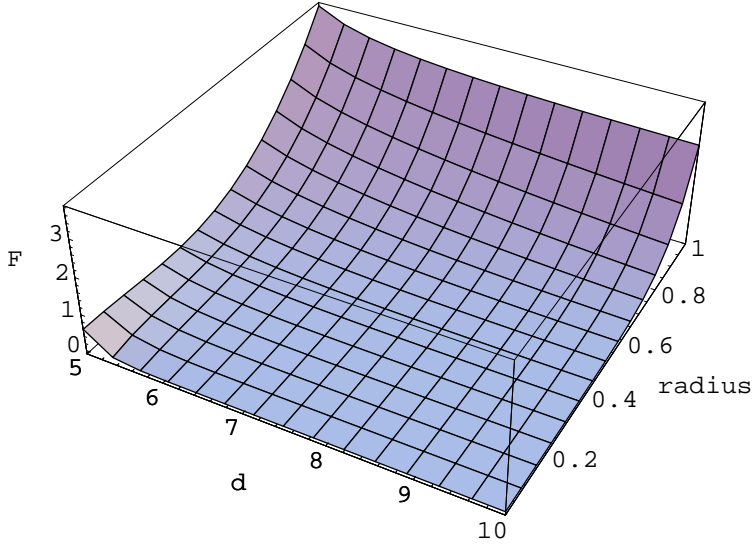


FIG. 3. The free energy of the Gauss-Bonnet black holes versus the horizon radius and space-time dimension with a fixed Gauss-Bonnet coefficient  $\tilde{\alpha}/l^2 = 0.2$ .

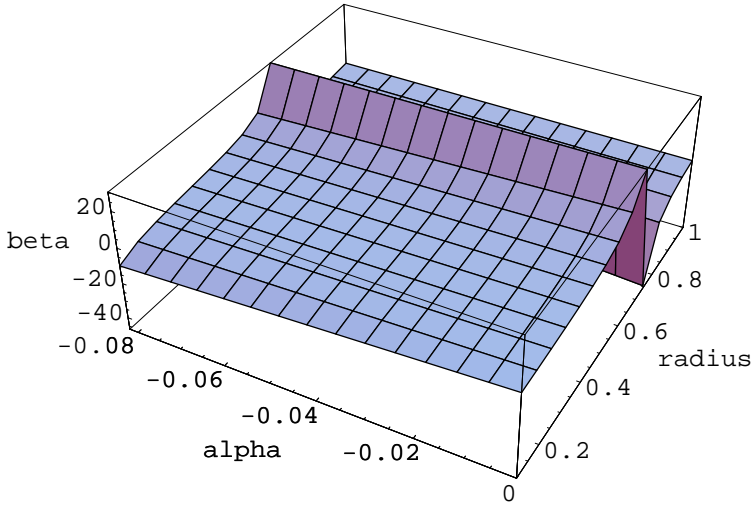


FIG. 4. The inverse temperature of the five dimensional Gauss-Bonnet black holes with  $-1/12 < \tilde{\alpha}/l^2 < 0$ .

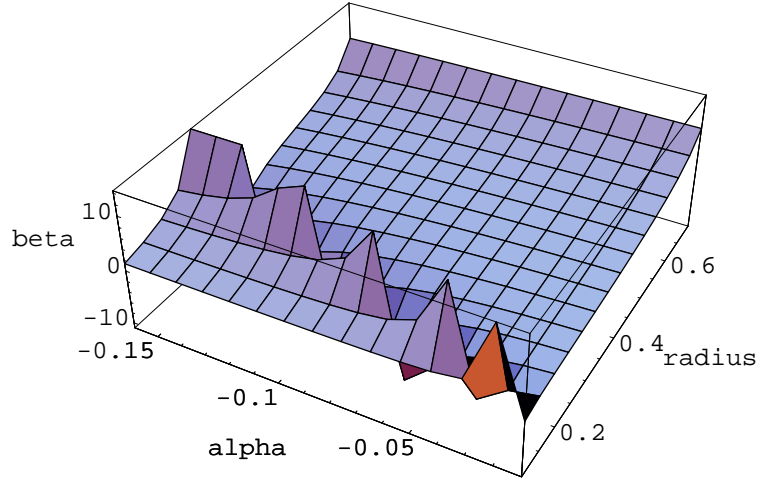


FIG. 5. The inverse temperature of the seven dimensional Gauss-Bonnet black holes with  $-17/100 < \tilde{\alpha}/l^2 < 0$ .

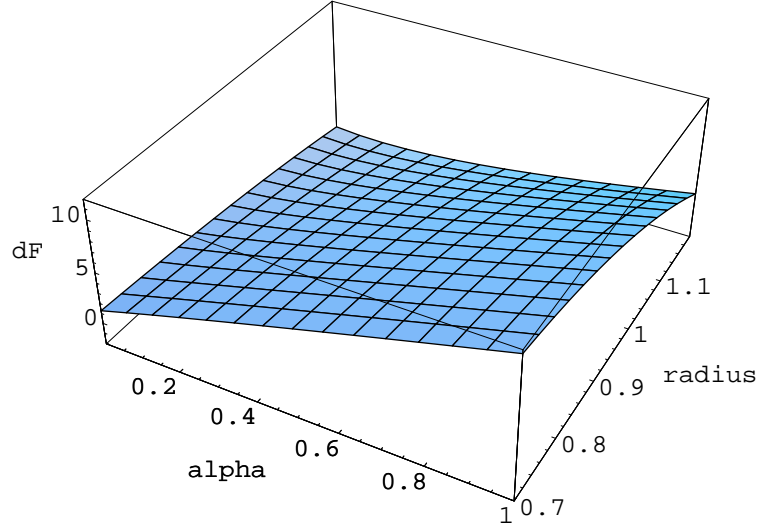


FIG. 6. The difference  $\Delta F$  of two free energies associated with the cosmological horizon for the case of five dimensions.

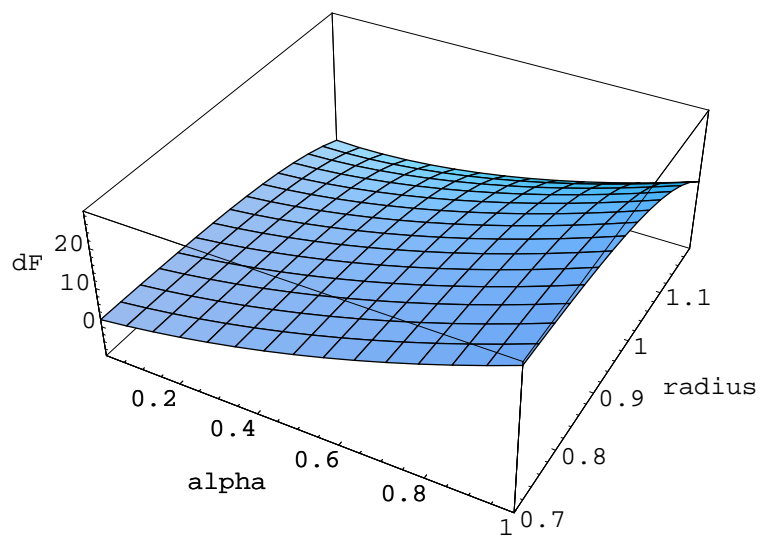


FIG. 7. The difference  $\Delta F$  of two free energies for the case of ten dimensions.

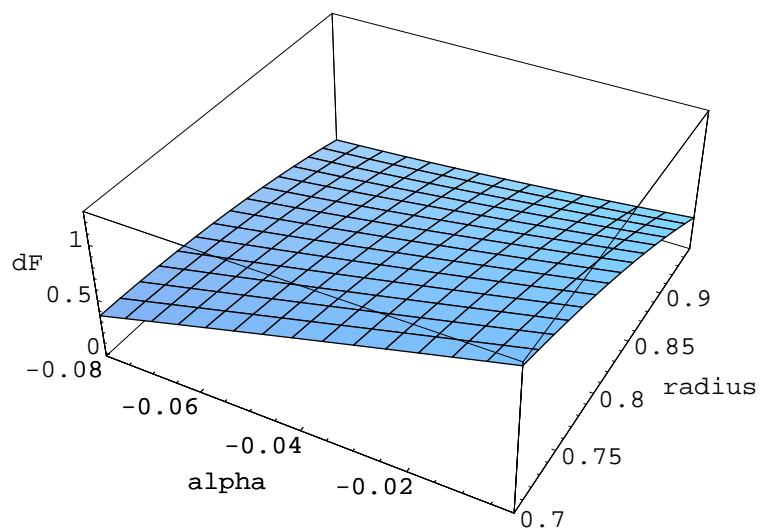


FIG. 8. The difference  $\Delta F$  of two free energies in the case of five dimensions with  $-1/12 < \tilde{\alpha}/l^2 < 0$ .